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Bi-Difference Sets, Order Relation, and Monoids

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Bi-difference sets generalize pseudo-difference sets and D-sets. Bi-difference sets automatically have an order relation if they are weaked slightly. As an application of the partially order relation, we present a characteristic of ideals in the weaked bi-difference sets. If a certain condition is satisfied then a bi-difference set becomes the union of monoids.

KEY WORDS: bi-difference sets; order; monoids.

1. INTRODUCTION

Dvurecenskij and Vetterlein in 2001 introduced an unsharp quantum logic structure and called it the *pseudo-effect algebra*, that is (Dvurecenskij and Vetterlein, 2001):

Let PE be a set with two special elements 0, 1, $\perp$ be a subset of $PE \times PE$, $\oplus : \perp \to PE$ be a binary operation, and the following axioms hold:

(PE1) $a \oplus b, (a \oplus b) \oplus c$ exist iff $b \oplus c, a \oplus (b \oplus c)$ exist, and in this case, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(PE2) For each $a \in PE$, there is exactly one $d \in PE$, and exactly one $e \in PE$ such that $a \oplus d = e \oplus a = 1$.
(PE3) If $a \oplus b$ exists, there are elements $d, e \in PE$ such that $a \oplus b = d \oplus a = b \oplus e$.
(PE4) If $1 \oplus a$ or $a \oplus 1$ exist, then $a = 0$.

Recently, Ma, Wu, and Lu introduced a new quantum logic structure and called it the *pseudo-difference set*, that is (Zhihao *et al.*, 2004):

A pseudo-difference poset is a partially ordered set (PD, $\leq$, 0, 1) with a maximum element 1 and a minimum element 0, two partial binary operations $\ominus_l$ and

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$\ominus_r$, and $b \ominus_l a$ are defined in PD iff $b \ominus_r a$ is defined in PD iff $a \leq b$ in PD, and the two operations $\ominus_l$ and $\ominus_r$ satisfy the following axioms:

(PD1) $b \ominus_l a \leq b$, $b \ominus_r a \leq b$.
(PD2) $b \ominus_l (b \ominus_r a) = a$, $b \ominus_r (b \ominus_l a) = a$.
(PD3) $(c \ominus_l b) \leq (c \ominus_l a)$, $(c \ominus_r b) \leq (c \ominus_r a)$.
(PD4) $(c \ominus_l a) \ominus_r (c \ominus_l b) = b \ominus_l a$, $(c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a$.
(PD5) If $1 \ominus_r (1 \ominus_l b \ominus_l a)$ is defined, then there exist $d, e \in PE$ such that

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) = (1 \ominus_r (1 \ominus_l a \ominus_l d)) = (1 \ominus_r (1 \ominus_l e \ominus_l b)).$$

If $1 \ominus_l (1 \ominus_r b \ominus_r a)$ is defined, then there exists $f, g \in PE$ such that

$$(1 \ominus_l (1 \ominus_r b \ominus_r a)) = (1 \ominus_l (1 \ominus_r a \ominus_r f)) = (1 \ominus_l (1 \ominus_r g \ominus_r b)).$$

Moreover, Ma, Wu, and Lu proved the following very important and interesting conclusion (Zhihao *et al.*, 2004):

Pseudo-difference posets and pseudo-effect algebras are the same thing.

On the other hand, Nanasiova in 1995 introduced the D-set and proved some important properties (Nanasiova, 1995).

In this paper, we introduce *bi-difference sets*, which depend mainly on the conditions (PD2) and (PD4) of pseudo-difference posets and generalize the D-sets, and prove some elementary properties of bi-difference sets. We prove also an important conclusion, that is, if the bi-difference sets are weaked slightly, then they have an order relation automatically. As an application of the partial order relation, we present a characteristic of ideals in the weaked bi-difference sets. Finally, we show that if a certain condition is satisfied, then a bi-difference set becomes the union of monoids.

2. BI-DIFFERENCE SETS

Definition 1. Let L be a nonempty set and $\ominus_l$, $\ominus_r$ be two partial binary operations on L. Then the set $(L, \ominus_l, \ominus_r)$ will be called a *bi-difference set* if the following conditions are satisfied:

(BD1) For any $a \in L$, $a \ominus_l a$ and $a \ominus_r a$ are defined and they are equal, denoted as $a \ominus_r a = a \ominus_l a = 0_a$.
(BD2) If $a \ominus_l b$ is defined, then $a \ominus_r (a \ominus_l b)$ is also defined and $a \ominus_r (a \ominus_l b) = b$; if $a \ominus_r b$ is defined, then $a \ominus_l (a \ominus_r b)$ is also defined, and $a \ominus_l (a \ominus_r b) = b$.
(BD3) If $a \ominus_l b$ and $b \ominus_l c$ are defined, then $(a \ominus_l c)$ is also defined, and $(a \ominus_l c) \ominus_r (a \ominus_l b) = (b \ominus_l c)$; if $a \ominus_r b$ and $b \ominus_r c$ are defined, then $(a \ominus_r c)$ is also defined and $(a \ominus_r c) \ominus_l (a \ominus_r b) = (b \ominus_r c)$.

Lemma 1. (Zhihao *et al.*, 2004). *If* $(L, \ominus_l, \ominus_r)$ *is a bi-difference set, then*

(BD4) $c \ominus_l a \ominus_r b = c \ominus_r b \ominus_l a$, $c \ominus_r a \ominus_l b = c \ominus_l b \ominus_r a$.
(BD5) $(c \ominus_l a) \ominus_l (b \ominus_l a) = (c \ominus_l b)$, $(c \ominus_r a) \ominus_r (b \ominus_r a) = (c \ominus_r b)$.

Theorem 1. *If $(L, \ominus_l, \ominus_r)$ is a bi-difference set, then*

(1) $a \ominus_l 0_a = a$, $a \ominus_r 0_a = a$, *for all* $a \in L$.
(2) $b \ominus_l a = 0_b$ *iff* $a = b$, $b \ominus_r a = 0_b$ *iff* $a = b$.
(3) $b \ominus_l a = b$ *iff* $a = 0_b$, $b \ominus_r a = b$ *iff* $a = 0_b$.
(4) *If* $c \ominus_l a \in L$, *then* $0_a = 0_c = 0_{c \ominus_l a}$. *If* $c \ominus_r a \in L$, *then* $0_a = 0_c = 0_{c \ominus_r a}$.
(5) *If* $c \ominus_l a = c \ominus_l b$, *then* $a = b$. *If* $c \ominus_r a = c \ominus_r b$, *then* $a = b$.
(6) *If* $a \ominus_l c = b \ominus_l c$, *then* $a = b$. *If* $a \ominus_r c = b \ominus_r c$, *then* $a = b$.
(7) *If* $c \ominus_r b, (c \ominus_r b) \ominus_l a \in L$, *then* $c \ominus_l a, (c \ominus_l a) \ominus_r b \in L$, *and*

$$c \ominus_l a \ominus_r b = c \ominus_r b \ominus_l a.$$

If $c \ominus_l b, (c \ominus_l b) \ominus_r a \in L$, *then* $c \ominus_r a, (c \ominus_r a) \ominus_l b \in L$, *and*

$$c \ominus_r a \ominus_l b = c \ominus_l b \ominus_r a.$$

(8) *If* $c \ominus_l a = d$, *then* $c \ominus_r d = a$. *If* $c \ominus_r a = d$, *then* $c \ominus_l d = a$.

Proof: We only prove the first part of each conclusions, since the second part of each conclusions can be obtained dually.

(1) Note that $a \ominus_l (a \ominus_r a) = a$, so $a \ominus_l 0_a = a$.
(2) If $a = b$, then $b \ominus_l a = 0_b$. If $b \ominus_l a = 0_b$, it follows from (BD2) and (1) that $a = b \ominus_r (b \ominus_l a) = b \ominus_r 0_b = b$.
(3) If $b \ominus_l a = b$, then $a = b \ominus_r (b \ominus_l a) = b \ominus_r b = 0_b$. The converse follows from (1) immediately.
(4) If $c \ominus_l a \in L$, then $(c \ominus_l a) \ominus_r (c \ominus_l a) \in L$, and $0_{c \ominus_l a} = (c \ominus_l a) \ominus_r (c \ominus_l a) = a \ominus_l a = 0_a$. On the other hand, note that $c \ominus_l a, c \ominus_l c \in L$, so it follows from (BD3) that $(c \ominus_l a) \ominus_r 0_c = (c \ominus_l a) \ominus_r (c \ominus_l c) = c \ominus_l a$, and so it follows from (3) that $0_{c \ominus_l a} = 0_c$.
(5) It follows from $(c \ominus_l a) \ominus_r (c \ominus_l b) = (b \ominus_l a) = 0_{c \ominus_l b} = 0_b$ and (2) that $a = b$.
(6) It follows from Lemma 1 and (4) that $(a \ominus_l c) \ominus_l (b \ominus_l c) = (a \ominus_l b) = 0_a$, so $a = b$.
(7) It follows from (BD4) of Lemma 1 immediately.
(8) If $c \ominus_l a \in L$ and $c \ominus_l a = d$, then $a = c \ominus_r (c \ominus_l a) = c \ominus_r d$. □

3. ORDER AND IDEALS OF BI-DIFFERENCE SETS

Now, we show that if the definition of the bi-difference sets is weakened slightly, then they have order relation automatically, for simple, we assume that for all $a \in L$, 0_a is same, that is,

Let $(L, \ominus_l, \ominus_r, 0)$, where $0 \in L$ be constant, and $\ominus_l, \ominus_r$ be partial binary operations on L satisfy:

(WBD1) For any $a \in L$, $a \ominus_l a$ and $a \ominus_r a$ are defined and $a \ominus_l a = 0 = a \ominus_r a$.
(WBD2) If $a \ominus_l b$ is defined, then $a \ominus_r (a \ominus_l b)$ is also defined and $(a \ominus_r (a \ominus_l b) \ominus_r b = 0$; if $a \ominus_r b$ is defined, then $(a \ominus_r (a \ominus_l b) \ominus_l b = 0$.
(WBD3) If $a \ominus_l b$ and $b \ominus_l c$ are defined, then $(a \ominus_l c)$ is also defined, and $((a \ominus_l c) \ominus_r (a \ominus_l b)) \ominus_l (b \ominus_l c) = 0$, $((a \ominus_l c) \ominus_r (a \ominus_l b)) \ominus_r (b \ominus_l c) = 0$; if $a \ominus_r b$ and $b \ominus_r c$ are defined, then $(a \ominus_r c)$ is also defined and $((a \ominus_r c) \ominus_l (a \ominus_r b)) \ominus_l (b \ominus_r c) = 0$, $((a \ominus_r c) \ominus_l (a \ominus_r b)) \ominus_r (b \ominus_r c) = 0$.
(WBD4) If $a \ominus_i b = b \ominus_j a = 0$, then we must have $a = b$, where $i, j = l, r$.
Then $(L, \ominus_l, \ominus_r, 0)$ is said to be a *weak bi-difference set*.

Now, we present two examples of weak bi-different sets:

Example 1. Let X be a nonempty set and its power set be denoted by $\mathcal{P}(X)$, $\emptyset$ be the empty set. Define $A \ominus_l B = A \ominus_r B = \emptyset$ if $A \subseteq B$, otherwise $A \ominus_l B = A \ominus_r B = A - B$.

Then it is easily to prove that $(\mathcal{P}(X), \ominus_l, \ominus_r, \emptyset)$ is a weak bi-different set.

Example 2. Let L be the set of all non-negative integers. If $a \leq b$, we define $a \ominus_l b = a \ominus_r b = 0$, otherwise we define $a \ominus_l b = a \ominus_r b = a - b$.

Then $(L, \ominus_l, \ominus_r, 0)$ is also a weak bi-different set.

From the definition of weak bi-different sets, we may prove:

(WBD5) If $a \ominus_l 0 = 0$, then $a = 0$; if $a \ominus_r 0 = 0$, then $a = 0$.
(WBD6) For any $a \in L$, $a = a \ominus_l 0 = a \ominus_r 0$.

Theorem 2. *Let $(L, \ominus_l, \ominus_r, 0)$ be a weak bi-difference set, $a, b \in L$. If we define a relation $\leq$ on $(L, \ominus_l, \ominus_r, 0)$ by $a \leq b$ iff $a \ominus_l b$ and $a \ominus_r b$ are defined and $a \ominus_l b = a \ominus_r b = 0$, then the relation $\leq$ is an order relation.*

Proof: It follows from $a \ominus_l a = a \ominus_r a = 0_a$ that $a \leq a$. □

If $a \leq b, b \leq a$, then $a \ominus_l b, a \ominus_r b, b \ominus_l a, b \ominus_r a$ are defined and they are all 0, so from (WBD4) that $a = b$.

If $a \leq b, b \leq c$, so $a \ominus_l b = 0, a \ominus_r b = 0, b \ominus_l c = 0, b \ominus_r c = 0$. On the other hand, it follows from (WBD3) and (WBD6) that $a \ominus_l c, a \ominus_r c$ are also 0, so $a \leq c$. Thus, $\leq$ is a order relation, the theorem is proved.

Now, by using the order relation of above, we present an interesting characteristic of ideals in the weak bi-difference set. At first, we need the following

(Meng and Jun, 1994):

Let $(L, \ominus_l, \ominus_r, 0)$ be a weak bi-difference set, I be a nonempty subset of L. If

(I1) $0 \in I$.
(I2) $x \ominus_l y \in I, y \in I$ imply $x \in I$, $x \ominus_r y \in I, y \in I$ imply $y \in I$.

Then I is said to be an *ideal* of $(L, \ominus_l, \ominus_r, 0)$.

Lemma 2. *Let I be an ideal of $(L, \ominus_l, \ominus_r, 0)$ and $x \in I$. If $y \leq x$, then $y \in I$.*

In fact, $y \leq x$ implies $y \ominus_l x = 0 \in I$. From $x \in I$ and (I2) that $y \in I$.

Theorem 3. *Let $(L, \ominus_l, \ominus_r, 0)$ be a weak bi-difference set, $I \subseteq L$, and $0 \in I$. Denote $A_l(x, y) = \{a : a \in L, a \ominus_l x \leq y\}$, $A_r(x, y) = \{a : a \in L, a \ominus_r x \leq y\}$. Then I is an ideal of $(L, \ominus_l, \ominus_r)$ iff for $\forall x, y \in I$, $A_l(x, y) \subseteq I$, $A_r(x, y) \subseteq I$.*

Proof: $\Rightarrow$. If $z \in A_i(x, y)$, $i = l, r$, we get $(z \ominus_i x) \leq y$, $y \in I$, $i = l, r$, it follows from Lemma 2 that $(z \ominus_i x) \in I$, $x \in I$, $i = l, r$, so by the definition of ideals that $z \in I$. □

$\Leftarrow$. If $A_i(x, y) \subseteq I, \forall x, y \in I, i = l, r$. Let $(z \ominus_i y) \in I$, $y \in I$, $i = l$ or $i = r$, note that $(z \ominus_r (z \ominus_l y)) \leq y$, $(z \ominus_l (z \ominus_r y)) \leq y$, so by the definition of A_l and A_r that $z \in A_i((z \ominus y), y) \subseteq I$, $i = l, r$, that is, I is an ideal of L.

4. BI-DIFFERENCE SETS AND MONOIDS

As we knew, the monoids is a very important algebra concept (Jacobson, 1974, p. 28). Now, we show that if a certain condition is satisfied, then each bi-difference set can become into the union of a family of monoids.

Lemma 3. *Let L be a bi-difference set, $a, b \in L$ and $0_b \ominus_r b, 0_b \ominus_l b \in L$. Then $a \ominus_l (0_b \ominus_r b) \in L$ iff $a \ominus_l b \in L$.*

Proof: If $a \ominus_l b \in L$, then $0_a = 0_b$. Note that $0_b \ominus_l (0_b \ominus_r b) = b$, we get $(a \ominus_r a) \ominus_l (0_b \ominus_r b) \in L$. Thus, it follows from Lemma 1 that $a \ominus_l (0_b \ominus_r b) \in L$.

If $a \ominus_l (0_b \ominus_r b) \in L$, it follows from Theorem 1 (4) that $(0_b \ominus_r b) \ominus_l 0_{0_b \ominus_r b} = (0_b \ominus_r b) \ominus_l 0_b \in L$, so by (BD3) we get $a \ominus_l 0_b \in L$. Note that $0_b \ominus_l b \in L$, it follows from (BD3) again that $a \ominus_l b \in L$. □

Now, we can define the partial operation $\oplus$ on the bi-difference set as follows:

If $0_b \ominus_r b \in L$, $0_b \ominus_l b \in L$, and $a \ominus_l b \in L$, then we define $a \oplus b := a \ominus_l (0_b \ominus_r b)$.

The following concepts are necessary in this section:

Definition 2. (Jacobson, 1974). A monoid is a triple $(M, p, 1)$ in which M is a nonempty set, p is an associative binary operation in M, and 1 is an element of M such that $p(1, a) = a = p(a, 1)$ for all $a \in M$, the element 1 is called the unit of $(M, p, 1)$.

Definition 3. A bi-difference set L is said to be a monoid bi-difference set if the following condition is satisfied:

(BD6) $a \ominus_l b \in L$ iff $b \ominus_r a \in L$. $a \ominus_r b \in L$ iff $b \ominus_l a \in L$.

Lemma 4. *If L is a monoid bi-difference set, then the following conclusions hold:*

(1) *For any $a \in L$, $0_a \ominus_l a \in L$, $0_a \ominus_r a \in L$.*
(2) *For $a, b \in L$, $a \ominus_l b \in L$ iff $a \ominus_r b \in L$ iff $0_a = 0_b$.*
(3) *If $a \ominus_l b \in L$, then $a \ominus_l b = 0_a \ominus_r (b \ominus_l a)$; if $a \ominus_r b \in L$, then $a \ominus_r b = 0_a \ominus_l (b \ominus_r a)$.*

Proof: We only prove the first part of each conclusion.

(1) Let $a \in L$. It follows from Theorem 1 that $a \ominus_r 0_a \in L$, note that from (BD6) we have $0_a \ominus_l a \in L$.
(2) It follows from Theorem 1 that if $a \ominus_l b \in L$, then $0_a = 0_b = 0_{a \ominus_l b}$. On the other hand, let $0_a = 0_b$. It follows from $a \ominus_l 0_a, 0_b \ominus_l b \in L$ and (BD3) that $a \ominus_l b \in L$.
(3) Let $a \ominus_l b \in L$. Then $0_a = 0_b$ and $0_a = a \ominus_l a = b \ominus_l b$. So $0_a \ominus_r (b \ominus_l a) = (b \ominus_l b) \ominus_r (b \ominus_l a) = a \ominus_l b$. □

Theorem 4. *If L is a monoid bi-difference set, then $(G(a), \oplus, 0_a) = \{b : b \in L$ and $0_b = 0_a\}$ is a monoid, 0_a is the unit element, and each $b \in L$ has the left inverse and the right inverse.*

Proof: It follows from Lemma 4 easily that $0_a \in G(a)$. If $b \in L$, then $b \oplus 0_a = b \ominus_l (0_a \ominus_r 0_a) = b \ominus_l (0_a) = b \ominus_l (0_b) = b$. Similar, we have $0_a \oplus b = b$. So 0_a is a unit element of $(G(a), \oplus, 0_a)$.

Now, we prove the associative law of the operation $\oplus$ as follows:

Let $a, b, c \in G(a)$. Then it follows easily from Lemma 3 and Lemma 4 that $(a \oplus b) \oplus c \in G(a)$. Denote $A = (a \oplus b) \oplus c = a \ominus_l (0_b \ominus_r b) \ominus_l (0_c \ominus_r c)$. It follows from Theorem 1 (7) that $A \ominus_r a = a \ominus_l (0_b \ominus_r b) \ominus_l (0_c \ominus_r c) \ominus_r a = 0_a \ominus_l (0_b \ominus_r b) \ominus_l (0_c \ominus_r c) = b \ominus_l (0_c \ominus_r c)$.

Denote $C = b \ominus_l (0_c \ominus_r c)$, $B = a \oplus (b \oplus c) = a \ominus_l (0_{b\oplus c} \ominus_r [b \ominus_l (0_c \ominus_r c)])$. Then it follows from $0_a = 0_{b\oplus c}$ that $B \ominus_r a = a \ominus_l (0_{b\oplus c} \ominus_r [b \ominus_l (0_c \ominus_r c)] \ominus_r a = 0_a \ominus_l (0_{b\oplus c} \ominus_r [b \ominus_l (0_c \ominus_r c)]) = b \ominus_l (0_c \ominus_r c) = C$.

So we get that $A = B$. Thus $(G(a), \oplus, 0_a)$ satisfies the associative law.

Let $b \in G(a)$ and denote $b^- := 0_a \ominus_l b$. Note that $b \oplus b^- = b \ominus_l [0_a \ominus_r (0_a \ominus_l b)] = 0_b = 0_a$, so the right inverse of $b \in G(a)$ is b^-. Similar, the left inverse of $b \in G(a)$ is $b^\sim := 0_a \ominus_r b$.

Thus, we proved that $(G(a), \oplus, 0_a)$ is a monoid and has the left inverse and right inverse for each $b \in G(a)$. □

The main result in this section is the following:

Theorem 5. *If L is a monoid bi-difference set, then L can be written as disjoint union of monoids* $\{(T_\alpha, \oplus_\alpha, 0_\alpha)\}_{\alpha\in\Gamma}$ *such that each element* $a \in (T_\alpha, \oplus_\alpha, 0_\alpha)$ *has the left inverse and the right inverse. Conversely, if L is the disjoint union of monoids* $\{(T_\alpha, \oplus_\alpha, 0_\alpha)\}_{\alpha\in\Gamma}$ *and each element* $a \in (T_\alpha, \oplus_\alpha, 0_\alpha)$ *has the left inverse and the right inverse, then L is a monoid bi-difference set.*

Proof: The first part of Theorem 5 follows from Theorem 4 immediately.

Conversely, let $\{(T_\alpha, \oplus_\alpha, 0_\alpha)\}_{\alpha\in\Gamma}$ be a family of disjoint monoids and for each element $a \in (T_\alpha, \oplus_\alpha, 0_\alpha)$ has the right inverse a^- and the left inverse $a^\sim$. Denote $T = \cup_{\alpha\in\Gamma} T_\alpha$.

First, we define a partial binary operation $\oplus$ on T:

$a \oplus b \in T$ iff there exists $\alpha \in \Gamma$ such that $a, b \in T_\alpha$ and define $a \oplus b = a \oplus_\alpha b$.

Next, let us define two partial binary operations $\ominus_l$ and $\ominus_r$ on T:

$a \ominus_l b \in T$ iff there exists $\alpha \in \Gamma$ with $a, b \in T_\alpha$ and $a \ominus_l b = a \oplus b^-$; $a \ominus_r b \in T$ iff there exists $\alpha \in \Gamma$ with $a, b \in T_\alpha$ and $a \ominus_r b = b^\sim \oplus a$.

Finally, we show that $(T, \ominus_r, \ominus_l)$ is a monoid bi-difference set.

If $a \in T$, then there is $\alpha \in \Gamma$ such that $a \in T_\alpha$. Note that $(T_\alpha, \oplus_\alpha, 0_\alpha)$ is a monoid with the left inverse and the right inverse for each element of $(T_\alpha, \oplus_\alpha, 0_\alpha)$, so we have $0_\alpha = a \oplus a^- = a^\sim \oplus a = a \ominus_l a = a \ominus_r a = 0_a \in T$. This showed that (BD1) holds and for each $a \in (T_\alpha, \oplus_\alpha, 0_\alpha)$, we have $0_a = 0_\alpha$.

If $a, b \in T$ and $a \ominus_l b$ is defined. It follows from the definitions of $a \ominus_l b$ and $(T_\alpha, \oplus_\alpha, 0_\alpha)$ that there is $\alpha \in \Gamma$ such that $a, b, a^-, b^-, a^\sim, b^\sim \in T_\alpha$ and $a \ominus_l b = a \oplus_\alpha b^-$, $b \ominus_l a = b \oplus_\alpha a^-$, $a^\sim \oplus_\alpha b = b \ominus_r a \in T_\alpha$. This showed that $b \ominus_r a \in T$. By using the same methods, we may prove that if $b \ominus_r a \in T$, then $a \ominus_l b \in T$. So (BD6) holds.

Let $a \ominus_l b \in T$. It follows from the proof process of above that there exists $\alpha \in \Gamma$ such that

$$(b^\sim \oplus_\alpha a) \oplus_\alpha (a^- \oplus_\alpha b) = 0_\alpha,$$

and

$$(a \oplus_\alpha b^\sim) \oplus_\alpha (b \oplus_\alpha a^-) = 0_\alpha.$$

So we get that:

$$(b^\sim \oplus_\alpha a)^- = a^- \oplus_\alpha b,$$
$$(b \oplus_\alpha a^-)^\sim = a \oplus_\alpha b^\sim.$$

These imply that

$$a \ominus_l (a \ominus_r b) = a \oplus_\alpha (b^\sim \oplus_\alpha a)^- = a \oplus_\alpha (a^- \oplus_\alpha b) = b,$$

and

$$a \ominus_r (a \ominus_l b) = b.$$

That is, (BD2) is proved.

Let $a \ominus_l b, b \ominus_l c \in T$. Then it is easy to prove that there exists $\alpha \in \Gamma$ such that $a, b, c \in T_\alpha$. Moreover,

$$\begin{aligned}(a \ominus_l c) \ominus_r (a \ominus_l b) &= (a \oplus_\alpha c^-) \ominus_r (a \oplus_\alpha b^-) = (a \oplus_\alpha b^-)^\sim \oplus_\alpha (a \oplus_\alpha c^-)\\ &= b \oplus_\alpha a^\sim \oplus_\alpha a \oplus_\alpha c^- = b \oplus_\alpha c^-\\ &= b \ominus_l c.\end{aligned}$$

Similarly, we get that

$$(a \ominus_r c) \ominus_l (a \ominus_r b) = b \ominus_r c.$$

Thus (BD3) is proved and T is a monoid bi-difference set. □

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